

# ON THE LAZAREV-LIEB EXTENSION OF THE HOBBY-RICE THEOREM

VERMONT RUTHERFOORD

**ABSTRACT.** O. Lazarev and E. H. Lieb proved that given  $f_1, \dots, f_n \in L^1([0, 1]; \mathbb{C})$ , there exists a smooth function  $\Phi$  that takes values on the unit circle and annihilates  $\text{span}\{f_1, \dots, f_n\}$ . We give an alternative proof of that fact that also shows the  $W^{1,1}$  norm of  $\Phi$  can be bounded by  $5\pi n + 1$ . Answering a question raised by Lazarev and Lieb, we show that if  $p > 1$  then there is no bound for the  $W^{1,p}$  norm of any such multiplier in terms of the norms of  $f_1, \dots, f_n$ .

The Hobby-Rice Theorem [HR] states

**Theorem 1.** *If  $f_1, \dots, f_n \in L^1([0, 1]; \mathbb{R})$  then there exists  $\Phi : [0, 1] \rightarrow \{-1, 1\}$  with at most  $n$  discontinuities such that for each  $k$*

$$\int_0^1 f_k(t) \Phi(t) dt = 0.$$

The theorem has applications in  $L^1$  approximation and in combinatorics, particularly necklace splitting problems [A]. An elegant proof of the Hobby-Rice Theorem was given by Pinkus [P] using the Borsuk-Ulam Theorem.

Motivated by a problem in mathematical physics, Lazarev and Lieb [LL] extended this result to obtain a smooth annihilator taking values on the unit circle, i.e.,

**Theorem 2.** *If  $f_1, \dots, f_n \in L^1([0, 1]; \mathbb{C})$  then there exists  $\theta \in C^\infty([0, 1]; \mathbb{R})$  such that*

$$(0.1) \quad \forall k, \quad \int_0^1 f_k(t) e^{i\theta(t)} dt = 0.$$

Lazarev and Lieb suggested that there should be a simpler proof, and in this spirit, we offer the following proof. They also raised the question of calculating the  $H^1 = W^{1,2}$  norm of  $f_k e^{i\theta}$ . Corollary 4 shows there is such  $\theta$  with  $\|e^{i\theta(\cdot)}\|_{W^{1,1}} \leq 5\pi n + 1$ . We also show that for  $p > 1$  there exists a large class of normed spaces  $\mathcal{N} = \{(N, \|\cdot\|_N)\}$  including  $L^1$  so that  $\|e^{i\theta(\cdot)}\|_{W^{1,p}}$  cannot be bounded by  $\|f_1\|_N, \dots, \|f_n\|_N$ .

*Proof of Theorem 2.* We may assume  $f_1, \dots, f_n$  are linearly independent in  $L^1$  and thus choose  $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = 1$  so that

$$M := \begin{bmatrix} f_1(t_1) & \dots & f_1(t_n) \\ \vdots & \ddots & \vdots \\ f_n(t_1) & \dots & f_n(t_n) \end{bmatrix}$$

is invertible and each  $t_j$  is a Lebesgue point of all  $f_k$  (Lemma 9).

For each  $u, v \in [-1, 1]$ , let  $\theta_{u+iv} : \mathbb{R} \rightarrow \mathbb{R}$  be a step function supported on and increasing in the interval  $[-1, 1]$  that takes the values  $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$  on intervals of lengths  $\frac{1+u}{2}, \frac{1+v}{2}, \frac{1-u}{2}, \frac{1-v}{2}$ , respectively. Thus  $\int_{-1}^1 e^{i\theta_{u+iv}(t)} dt = u + iv$ .

Choose  $\psi \in C^\infty(\mathbb{R}; \mathbb{R}^+)$  supported on  $[-1, 1]$  such that  $\int \psi(t) dt = 1$ . Let  $\psi_h(t) = \psi(t/h)/h$ .

Also let  $I_S$  be the indicator function of  $S$ . Define  $\theta_{h,z}^\# = \theta_z I_{(h-1, 1-h)} + 2\pi I_{[1-h, \infty)}$  and

$$\theta_{h,z} = \begin{cases} \theta_z & \text{if } h = 0 \\ \psi_h * \theta_{h,z}^\# & \text{if } 0 < h < 1 \end{cases}$$

Note that if  $h > 0$  then  $\theta_{h,z}(-1) = 0$  and  $\theta_{h,z}(1) = 2\pi$ , while  $\theta_{h,z}^{(m)}(\pm 1) = 0$  for all  $m \geq 1$ .

Define  $D = \{z \in \mathbb{C} : |z| \leq 1\}$ ,  $d = \min_{j \in \{0 \dots n\}} (t_{j+1} - t_j)/2$ , and  $Q : [0, d] \times D^n \rightarrow \mathbb{C}^n$ :

$$Q(h; \vec{z}) = \begin{cases} \left( \sum_{j=1}^n z_j \cdot f_k(t_j) \right)_{k=1 \dots n} & \text{if } h = 0 \\ \left( \sum_{j=1}^n \int_{-1}^1 f_k(th + t_j) e^{i\theta_{h,z_j}(t)} dt \right)_{k=1 \dots n} & \text{if } 0 < h \leq d \end{cases}$$

with  $\vec{z} := (z_1, \dots, z_n)$ . Since  $Q(0; \vec{z}) = M(\vec{z})$ , Lemma 10 shows there is  $\delta \in (0, d]$  such that for all  $\vec{z} \in D^n$

$$\vec{z} - M^{-1}(Q(\delta; \vec{z})) \in \frac{1}{2}D^n.$$

Let  $L_\delta = [0, 1] \setminus \bigcup_{j=1}^n (t_j - \delta, t_j + \delta)$ . By applying the Hobby-Rice Theorem<sup>1</sup> to  $f_1 I_{L_\delta}, \dots, f_n I_{L_\delta}$  and smoothing a finite set of discontinuities, we obtain  $\phi \in C^\infty([0, 1]; \mathbb{R})$  supported on  $L_\delta$  so that

$$\vec{r} := \left( \int_{L_\delta} f_k(t) e^{i\phi(t)} dt \right)_{k=1 \dots n} \in \frac{\delta}{2} M(D^n).$$

Since  $\phi$  vanishes together with all its derivatives at all  $t_j \pm \delta$ , for all  $\vec{z} \in D^n$

$$\theta_{\vec{z}}^*(t) = \begin{cases} \theta_{\delta, z_j}((t - t_j)/\delta) + 2\pi(j - 1) & \text{if } t \in [t_j - \delta, t_j + \delta] \text{ and } 1 \leq j \leq n \\ \phi(t) & \text{if } t \in [0, t_1 - \delta) \\ \phi(t) + 2\pi j & \text{if } t \in (t_j + \delta, t_{j+1} - \delta) \text{ and } 1 \leq j \leq n - 1 \\ \phi(t) + 2\pi n & \text{if } t \in (t_n + \delta, 1] \end{cases}$$

is in  $C^\infty([0, 1]; \mathbb{R})$ . Lemma 11 establishes the continuity of

$$T(\vec{z}) := \left( \int_0^1 f_k(t) e^{i\theta_{\vec{z}}^*(t)} dt \right)_{k=1 \dots n} = \delta Q(\delta; \vec{z}) + \vec{r}.$$

Since  $\vec{z} - M^{-1}(Q(\delta; \vec{z}))$  and  $\frac{1}{\delta} M^{-1}(\vec{r})$  are in  $\frac{1}{2}D^n$  for all  $\vec{z} \in D^n$ , then  $\vec{z} - \frac{1}{\delta} M^{-1}(T(\vec{z})) \in D^n$ . By Brouwer's Fixed Point Theorem, there exists  $\vec{z}_0 \in D^n$  such that  $\vec{z}_0 - \frac{1}{\delta} M^{-1}(T(\vec{z}_0)) = \vec{z}_0$ , that is to say  $T(\vec{z}_0) = 0$ .  $\square$

<sup>1</sup>The Riemann-Lebesgue Lemma also suffices, but the Hobby-Rice Theorem enables us to compute a bound of the  $W^{1,1}$  norm in Corollary 4.

**Definition 3.** Let

$$\begin{aligned} \|g(\cdot)\|_{W^{1,p}} &= \left( \int_0^1 |g(t)|^p dt + \int_0^1 |g'(t)|^p dt \right)^{\frac{1}{p}} \\ \text{and } \|g(\cdot)\|_{\overset{\circ}{W}^{1,p}} &= \left( \int_0^1 |g'(t)|^p dt \right)^{\frac{1}{p}}. \end{aligned}$$

**Corollary 4.** If  $f_1, \dots, f_n \in L^1([0, 1]; \mathbb{C})$  then there exists  $\theta \in C^\infty([0, 1]; \mathbb{R})$  such that for each  $k$

$$\int_0^1 f_k(t) e^{i\theta(t)} dt = 0$$

and  $\|e^{i\theta(\cdot)}\|_{W^{1,1}} \leq 5\pi n + 1$ .

*Proof.* The calculation of the bound follows from a careful selection of  $\phi$  in the preceding proof. The Hobby-Rice Theorem applied to the  $n$  real parts and  $n$  imaginary parts of  $f_k I_{L_\delta}$  implies that there exists  $\phi^\# : \mathbb{R} \rightarrow \{0, \pi\}$  with at most  $2n$  discontinuities such that for each  $k$

$$\int_0^1 f_k(t) I_{L_\delta} e^{i\phi^\#(t)} dt = 0.$$

Since this equation still holds if  $\phi^\#$  is replaced with  $\pi - \phi^\#$ , choose such  $\phi^\#$  that is non-zero on at most  $n$  points at the boundary of  $L_\delta$ . Note that if  $\phi^\# I_{L_\delta}$  is discontinuous at  $t$ , then either  $\phi^\#$  is discontinuous at  $t$  or  $t = t_j \pm \delta$  where  $1 \leq j \leq n$  and  $\phi^\#(t) \neq 0$ . Thus  $\phi^\# I_{L_\delta}$  has at most  $3n$  discontinuities. Choose  $\eta > 0$  so that by selecting

$$\phi = (\phi^\# I_{L_{\delta+\eta}}) * \psi_\eta$$

then

$$\vec{r}_\eta := \left[ \int_{L_\delta} f_k(t) e^{i\phi(t)} dt \right]_{k=1 \dots n} \in \frac{\delta}{2} M(D^n).$$

Note that  $\phi$  vanishes together with all its derivatives at all  $t_j \pm \delta$ . Also note that  $\phi^\# I_{L_{\delta+\eta}}$  has no more discontinuities than  $\phi^\# I_{L_\delta}$ , which is at most  $3n$ . Thus there exist  $m \leq 3n$  and  $0 < y_1 < \dots < y_m < 1$  such that  $\phi^\# I_{L_{\delta+\eta}}(t)$  or  $\pi - \phi^\# I_{L_{\delta+\eta}}(t)$  for all  $t \in [0, 1] \setminus \{y_1, \dots, y_m\}$  is equal to  $\pi \sum_{j=1}^m (-1)^{j+1} I_{[y_j, \infty)}(t)$ . Consequently,

$$\begin{aligned} \int_{L_\delta} |(\theta_z^*)'(t)| dt &= \int_{L_\delta} |\phi'(t)| dt \\ &= \int_0^1 |\phi'(t)| dt \\ &= \int_0^1 |(\phi^\# I_{L_{\delta+\eta}} * \psi_\eta)'(t)| dt \\ &\leq \pi \sum_{j=1}^m \int_0^1 (I_{[y_j, \infty)} * \psi_\eta)'(t) dt \\ &\leq 3\pi n \end{aligned}$$

Recall that  $\theta_{\delta,z}$  is an increasing function with  $\theta_{\delta,z}(-1) = 0$  and  $\theta_{\delta,z}(1) = 2\pi$  for all  $z \in D$ . Thus  $\int_{t_j-\delta}^{t_j+\delta} |(\theta_z^*)'(t)| dt = 2\pi$  for  $1 \leq j \leq n$  and so  $\int_0^1 |(\theta_z^*)'(t)| dt \leq 5\pi n$ . Consequently,  $\|e^{i\theta_z^*(\cdot)}\|_{\overset{\circ}{W}^{1,1}} \leq 5\pi n$  and  $\|e^{i\theta_z^*(\cdot)}\|_{W^{1,1}} \leq 5\pi n + 1$ . Since  $\max_{t \in [0,1]} |\theta_z^*(t)| \leq (2n+1)\pi$ , it follows that  $\|\theta_z^*(\cdot)\|_{W^{1,1}} \leq (7n+1)\pi$ .  $\square$

Clearly, if  $f_1, \dots, f_n$  are real valued, they may be combined into  $\lceil \frac{n}{2} \rceil$  complex valued functions and the bounds reduce accordingly.

For  $p > 1$  the situation is different.

**Definition 5.** Let

$$A(f) = \left\{ \theta \in C^\infty([0, 1]; \mathbb{R}) : \int_0^1 f(t) e^{i\theta(t)} dt = 0 \right\}.$$

**Definition 6.** Let

$$\rho(f) = \inf \left\{ \int_0^1 |\theta'(t)|^p dt : \theta \in A(f) \right\}.$$

**Definition 7.** Let

$$(\Upsilon_n f)(t) = \begin{cases} f(2^n t) & \text{if } 0 \leq t \leq 2^{-n} \\ 0 & \text{otherwise} \end{cases}.$$

**Theorem 8.** Assume  $N$  is a norm for which there exists  $f \in L^1([0, 1]; \mathbb{C})$  such that  $0 < \|\Upsilon_n f(\cdot)\|_N < \infty$  for all  $n \geq 1$  and  $\rho(f) > 0$ .

Then, given any  $l, K \in \mathbb{R}^+$ , there exists  $g \in L^1([0, 1]; \mathbb{C})$  such that  $\|g(\cdot)\|_N = l$  and  $\rho(g) > K$ .

*Proof.* Choose  $\epsilon > 0$  and  $\theta \in A(\Upsilon_n f)$  such that  $\int_0^1 |\theta'(t)|^p dt < \rho(\Upsilon_n f) + \epsilon$ . Then  $(\Upsilon_{-n}\theta)|_{[0,1]} \in A(f)$ , and so

$$\begin{aligned} \rho(\Upsilon_n f) + \epsilon &> \int_0^1 |\theta'(t)|^p dt \\ &\geq \int_0^{2^{-n}} |\theta'(t)|^p dt \\ &= 2^{-n} \int_0^1 |\theta'(2^{-n}t)|^p dt \\ &= 2^{n(p-1)} \int_0^1 |(\theta(2^{-n}t))'|^p dt \\ &\geq 2^{n(p-1)} \rho(f) \end{aligned}$$

proving  $\rho(\Upsilon_n f) \geq 2^{n(p-1)} \rho(f)$ .

Also, since  $A(g) = A(cg)$  for all  $c \neq 0$  then

$$2^{n(p-1)} \rho(f) \leq \rho(\Upsilon_n f) = \rho(l \Upsilon_n f / \|(\Upsilon_n f)(\cdot)\|_N).$$

Consequently, if  $n$  is large enough so that  $2^{n(p-1)} \rho(f) > K$  then  $g := l \Upsilon_n f / \|(\Upsilon_n f)(\cdot)\|_N$  has the property that  $\rho(g) > K$  and  $\|g(\cdot)\|_N = l$ .  $\square$

The  $W^{1,p}$  norms of  $f_k(t) e^{i\theta(t)}$  fare no better, since if  $f_1(t) = 1$ , then  $\|f_1(\cdot) e^{i\theta(\cdot)}\|_{W^{1,p}} = \|e^{i\theta(\cdot)}\|_{W^{1,p}} \geq (\max_k \rho(f_k))^{\frac{1}{p}}$ .

#### LEMMAS

We include the lemmas that were used above, some or all of which may be familiar to the reader.

**Lemma 9.** *If  $f_1, \dots, f_n \in L^1([0, 1]; \mathbb{C})$  are linearly independent in  $L^1$ , then there exist  $t_1, \dots, t_n \in (0, 1)$  so that*

$$M := \begin{bmatrix} f_1(t_1) & \dots & f_1(t_n) \\ \vdots & \ddots & \vdots \\ f_n(t_1) & \dots & f_n(t_n) \end{bmatrix}$$

*is invertible and  $t_j$  is a Lebesgue point of  $f_k$  for each  $j, k \in 1 \dots n$ .*

*Proof.* Let  $P$  be the set of points in  $(0, 1)$  that are Lebesgue points for all  $f_k$ .

The case  $n = 1$  is clear. If  $n > 1$ , let us assume inductively that there are  $t_1, \dots, t_{n-1} \in P$  such that  $M' := [f_k(t_j)]_{(n-1) \times (n-1)}$  is invertible. Thus there exist  $\beta_1, \dots, \beta_{n-1} \in \mathbb{C}$  such that

$$[\beta_1 \dots \beta_{n-1}] M' = [f_n(t_1) \dots f_n(t_{n-1})].$$

Furthermore, since  $f_1, \dots, f_n$  are linearly independent in  $L^1$ , there exists  $t_n \in P$  such that

$$y_n := f_n(t_n) - \beta_1 f_1(t_n) - \dots - \beta_{n-1} f_{n-1}(t_n) \neq 0.$$

Thus  $M := [f_k(t_j)]_{n \times n}$  has a nonzero determinant, namely  $y_n \det M'$ . □

**Lemma 10.** *If  $t_0 \in (0, 1)$  is a Lebesgue point of  $f \in L^1([0, 1]; \mathbb{C})$ , then, uniformly in  $z \in D$ ,*

$$\lim_{h \rightarrow 0^+} \int_{-1}^1 f(th + t_0) \cdot \theta_{h,z}(t) dt = f(x) \cdot z$$

*Proof.* Given  $\epsilon > 0$ , let  $\delta_1 < \epsilon/20\pi(|f(t_0)| + 1)$ . If  $0 < h < \delta_1$ , then, since  $\theta_z(t)$  is a step function, for all values of  $t \in (2h - 1, 1 - 2h)$  that are not within distance  $h$  of a discontinuity of  $\theta_z$ ,  $\theta_{h,z}(t) = \theta_z(t)$ . Since there are at most three discontinuities of  $\theta_z$  in  $(2h - 1, 1 - 2h)$  and  $\theta_{h,z}(t), \theta_z(t) \in [0, 2\pi]$ ,

$$2\pi \cdot 6h \geq \int_{2h-1}^{1-2h} |\theta_{h,z}(t) - \theta_z(t)| dt$$

and so

$$\begin{aligned} \int_{-1}^1 \left| e^{i\theta_{h,z}(t)} - e^{i\theta_z(t)} \right| dt &\leq \int_{-1}^1 |\theta_{h,z}(t) - \theta_z(t)| dt \\ &\leq 2\pi \cdot (6 + 4)h \\ &< \epsilon / (|f(t_0)| + 1). \end{aligned}$$

Choose  $\delta_2 > 0$  so that if  $0 < h < \delta_2$  then

$$\int_{-1}^1 |f(th + t_0) - f(t_0)| dt < \epsilon/2$$

and let  $\delta = \min \{\delta_1, \delta_2\}$ . Then

$$\begin{aligned}
& \left| \int_{-1}^1 f(th + t_0) \cdot e^{i\theta_{h,z}(t)} dt - \overbrace{\int_{-1}^1 f(t_0) \cdot e^{i\theta_z(t)} dt}^{=f(t_0) \cdot z} \right| \\
& \leq \left| \int_{-1}^1 (f(th + t_0) - f(t_0)) e^{i\theta_{h,z}(t)} dt \right| + \left| \int_{-1}^1 f(t_0) (e^{i\theta_{h,z}(t)} - e^{i\theta_z(t)}) dt \right| \\
& \leq \int_{-1}^1 |f(th + t_0) - f(t_0)| dt + |f(t_0)| \int_{-1}^1 |e^{i\theta_{h,z}(t)} - e^{i\theta_z(t)}| dt \\
& < \epsilon
\end{aligned}$$

□

**Lemma 11.** *If  $f \in L^1([0, 1]; \mathbb{C})$ ,  $t_0 \in (0, 1)$ , and  $0 < h < \min \{t_0, 1 - t_0\}$ , then*

$$q(z) := \int_{-1}^1 f(th + t_0) e^{i\theta_{h,z}(t)} dt$$

is a continuous function of  $z \in D$ .

*Proof.* Since  $\theta_{h,z}^\#$  is a step function whose intervals of constancy vary continuously with  $z$ , for  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $|z_1 - z_2| < \delta$  then

$$\int_{-1}^1 \left| \theta_{h,z_1}^\#(t) - \theta_{h,z_2}^\#(t) \right| dt < \epsilon / (\|f\|_{L^1} + 1) h \|\psi(t)\|_{L^\infty}.$$

Then

$$\begin{aligned}
\left\| e^{\theta_{h,z_1}(t)} - e^{\theta_{h,z_2}(t)} \right\|_{L^\infty} & \leq \left\| \theta_{h,z_1}(t) - \theta_{h,z_2}(t) \right\|_{L^\infty} \\
& < \epsilon / (\|f\|_{L^1} + 1)
\end{aligned}$$

and so

$$\begin{aligned}
|q(z_1) - q(z_2)| & \leq \int_{-1}^1 \left| f(th + t_0) (e^{i\theta_{h,z_1}(t)} - e^{i\theta_{h,z_2}(t)}) \right| dt \\
& < \epsilon.
\end{aligned}$$

□

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VERMONT RUTHERFOORD, DEPARTMENT OF MATHEMATICAL SCIENCES, FLORIDA ATLANTIC UNIVERSITY, 777 GLADES ROAD, BOCA RATON, FL 33431, USA

*E-mail address:* [vermont.rutherford@gmail.com](mailto:vermont.rutherford@gmail.com)